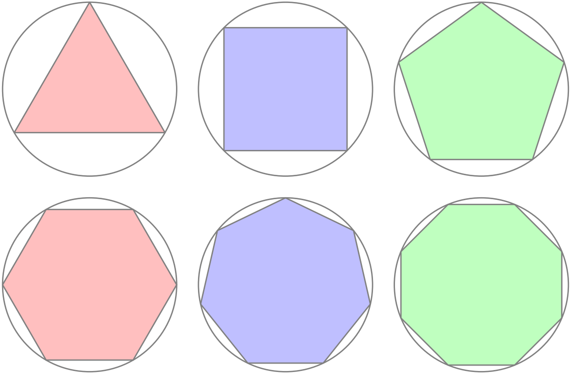
Foundations of computer science and numerical methods Laboratory of Ruby Method of exhaustion: Archimedes, the circle and π

Martin Brugnara[1] Marco Frego[2]

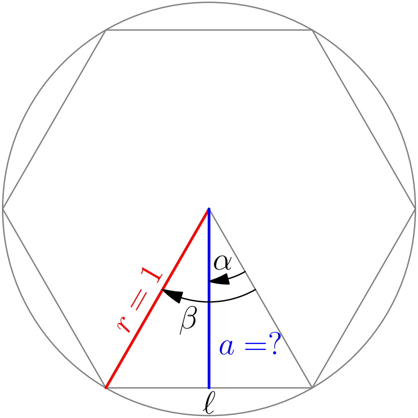
[1]mb@disi.unitn.eu (University of Trento)
disi.unitn.eu/~brugnara
[2]marco.frego@unitn.it (University of Trento)

Method of Exhaustion for the Area of the Circle

Exhaustion is a way to compute the area of a circle by inscribing inside a regular polygon of increasing area and number of sides. As the area of the polygon increases, as well as the number of sides, it approaches the the area π of the unitary circle.



Starting from the unitary circle, we need to find the apothem *a* of the inscribed polygon of *n* sides:



$$\beta = \frac{2\pi}{n}$$

$$\alpha = \frac{\beta}{2} = \frac{\pi}{n}$$

$$\frac{\ell}{2} = r\sin\alpha = \sin\frac{\pi}{n}$$

$$a = r\cos\alpha = \cos\frac{\pi}{n}$$

Computing the area

A quantity which is constant for all the regular polygons with n sides is the ration between the apothem and the side. It is called fix number as it is independent of r and is denoted by

$$f_n := \frac{a}{\ell} = \frac{r\cos\frac{\pi}{n}}{2r\sin\frac{\pi}{n}} = \frac{1}{2\tan\frac{\pi}{n}}$$

Some results from the high school:

$$f_3 = \frac{\sqrt{3}}{6} \approx 0.289 \qquad f_5 = \frac{\sqrt{2}}{40}(5+\sqrt{5})^{\frac{3}{2}} \approx 0.688$$

$$f_4 = \frac{1}{2} = 0.5 \qquad f_6 = \frac{\sqrt{3}}{2} \approx 0.866$$

Computing the area

The area of the polygon is simply the area of one of triangles that have as base the side ℓ and height the apothem, multiplied by the number of sides n, hence

$$A_n = \frac{\ell a}{2}\, n = \frac{2r\sin\frac{\pi}{n} r\cos\frac{\pi}{n}}{2}\, n\,\bigg|_{r=1} = n\sin\frac{\pi}{n}\cos\frac{\pi}{n}.$$

With the standard trigonometric identity $\sin 2x = 2\sin x\cos x$, we can further simplify the previous expression as

$$A_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right).$$

Another constant quantity

It can be observed as a side result, that for a regular polygon, the ratio between the area and the square of the side is constant, e.g.

$$k_n := \frac{A_n}{\ell^2}.$$

To prove this, simply expand the above line using the explicit expressions for the area and the side given in the previous pages:

$$k_n = \frac{\ell a n}{2} \cdot \frac{1}{\ell^2} = \frac{a}{\ell} \cdot \frac{n}{2} = f_n \frac{n}{2}.$$

Proof of the exhaustion

The last step is to prove that the limit of the area A_n for $n \to \infty$ converges to π, in other words the inscribed polygons 'fill completely' the circle. This can be done solving the limit:

$$\lim_{n \to +\infty} A_n = \lim_{n \to +\infty} \frac{n}{2} \sin\left(\frac{2\pi}{n}\right)$$

Being an indeterminate form, with the change of variable $m = 1/n$ we transform the limit into

$$\lim_{m \to 0} \frac{1}{2m} \sin(2\pi m),$$

which is a $0/0$ case that can be recast into the form $\sin(x)/x$ by:

$$\lim_{m \to 0} \pi \frac{\sin(2\pi m)}{2\pi m} = \pi \cdot 1 = \pi.$$

Some numerical results

The computation in Ruby of the area A_n gives the following results:

$$
\begin{aligned}
A_3 &= 1.299038106 \\
A_{10} &= 2.938926262 \\
A_{100} &= \mathbf{3.1}39525977 \\
A_{1\,000} &= \mathbf{3.1415}71983 \\
A_{10\,000} &= \mathbf{3.141592}448 \\
A_{100\,000} &= \mathbf{3.14159265}2 \\
&\vdots \\
\pi &= \mathbf{3.141592654}
\end{aligned}
$$